Statistical mechanics of the shallow water system

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We extend the formalism of statistical mechanics developed for the two-dimensional (2D) Euler equation to the case of shallow water system. Relaxation equations towards the maximum entropy state are proposed, which provide a parametrization of subgrid-scale eddies in 2D compressible turbulence.

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I. INTRODUCTION

Two-dimensional flows at high Reynolds numbers have the striking property of organizing spontaneously into large-scale coherent vortices [1,2]. These vortices are ubiquitous in geophysical and astrophysical flows with the well-known example of jupiter's great red spot, a huge vortex persisting for more than three centuries in a turbulent shear. They share some common features with stellar systems, such as, elliptical or spherical galaxies that form after a phase of ''violent relaxation'' [3–5]. They can also have applications in the process of planet formation, which may have begun inside persistant gaseous vortices born out of the protoplanetary nebula [6–9]. Understanding and predicting the structure of these organized states is still a challenging problem.

Since the dynamics of these systems is highly nonlinear, a deterministic description of the flow for late times is impossible and one must take recourse to statistical methods. The self-organization was first (qualitatively) explained by Onsager [10], using equilibrium statistical mechanics of singular point vortices. Explicit results have been obtained by Montgomery [11], Joyce and Montgomery [12], and Lundgren and Pointin [13] using a mean field approximation for this point vortex system. See Ref. [14] for a good review of the early statistical approaches of two-dimensional (2D) turbulence. The case of nonsingular vorticity fields has been later on treated by Kuzmin [15], Miller [16], and Robert and Sommeria [17]. The resulting equilibrium states are qualitatively similar to the mean field results [11–13] of the point vortex statistics, but there are quantitative differences. To apply this theory, the initial vorticity probability distribution has to be discretized in vorticity levels, and each of the corresponding vorticity patches is supposed to mix in the final self-organized state. We shall, therefore, call this approach the vortex patch statistical mechanics, although it applies to any continuous initial vorticity field: there is a well-defined limit as the discretization in vorticity levels is refined. This is unlike a discretization in terms of singular point vortices, whose statistical equilibrium depends on an arbitrary distribution of point vortex strengths.

While the point vortices form an N-body hamiltonian system, with firmly rooted statistical mechanics, the justification of the vortex patch statistical mechanics is more elusive. Nevertheless it has the advantage of providing a systematic framework to tackle self-organization in 2D flows, with well-defined predictions. At a microscopic level, complex, inviscid, deformation of the vorticity field creates an intricate filamentation; however, if we introduce a macroscopic level of description (a ''coarse graining'') it can be shown that an overwhelming majority of these microscopic configurations are close to a macroscopic state (the statistical equilibrium or Gibbs state) obtained by maximizing a ''mixing entropy'' while accounting for all the constraints of the Euler equation (the conservation of energy and of the global probability distribution of vorticity levels). The resulting Gibbs state turns out to be a steady solution of the Euler equations with superimposed small-scale fluctuations. Although these fluctuations are eventually smoothed out by viscosity or other small-scale effects (e.g., 3D turbulence in geophysical applications), the organization is controlled by the inertial stirring process.

The validity of the prediction, therefore, relies on the assumption that the inertial organization occurs faster than the change of vorticity levels due to viscosity (or other small-scale smoothing process), whose effect is enhanced by the concomitant straining of vorticity structures. Comparisons of the predicted Gibbs states with the results of numerical simulations at high Reynolds numbers provide good quantitative agreement when stirring is sufficiently rapid to yield equilibrium before significant influence of viscous effects [18–20]. In contrast, discrepancies appear when the final organization occurs after a long evolution time [19,21]. Similar conclusions have been drawn by comparing statistical mechanics predictions with laboratory experiments [22]. The theory is particularly useful to interpret electron plasma experiments [23], whose dynamics is well described by the 2D Euler equation (with no viscosity in the standard sense). Modifications of the theory have been proposed in attempts to keep only the conservation laws that are the most robust in the presence of weak viscous effects [24]. Note that although these different statistical mechanics approaches bear quantitative differences, they share an essential common property: they all predict a self-organization into a steady state with a monotonic relationship between vorticity and stream function.

For applications to atmospheric or oceanic motion, the Coriolis force and density stratification have to be taken into account. A first step in that direction is provided by the quasigeostrophic model. The application of the vortex patch statistical mechanics to this case is formally straightforward as the flow is still assumed to be nondivergent, and the vorticity is just replaced by a potential vorticity [25–28]. In the presence of planetary curvature or topography (beta effect), the point vortices are no more solutions of the system, but another approach, using a truncated spectral model [29] had been previously applied [30,31]. It provides a rationale for understanding the eddy generation of mean currents over the sloping margins of the oceans [32]. The resulting mean flow is recovered as a particular limit of the vortex patch equilibrium states [27].

The quasigeostrophic model is, however, of limited applicability: the shallow water system, dynamically equivalent to a compressible 2D flow, provides a more accurate and general description of geophysical flows [33]. It can be directly generalized into multilayer models used as operational predicting tools for oceanography [34]. Several numerical computations, for instance Refs. [35–37], indicate inertial organization of vortical motion into coherent vortices, such as with the incompressible Euler equations. However, none of the statistical mechanics approaches have been previously extended to the shallow water system. We propose here an attempt in this direction, stating the formalism for the generalization of the vortex patch statistical mechanics.

This statistical mechanics is closely related to the concept of potential vorticity (PV) mixing, widely used to interpret oceanic or atmospheric data: for instance, the circulation and isotherm shape in Atlantic gyres can be predicted by an assumption of uniform PV [38]. The idea of PV uniformization has been used also to interpret features of atmospheric winds in jupiter's atmosphere [39]. However, PV mixing is necessarily limited to some regions, in particular, due to the constraints on energy conservation. The statistical mechanics quantifies this idea by predicting the most mixed state consistent with energy conservation. In a shallow water rotating system, it yields zones of uniform PV separated by intense jets (scaling as the Rossby radius of deformation). This structure is illustrated in Fig. 1 (taken from Bouchet and Dumont [40]) showing a model of the great red spot of jupiter. The present paper provides the formalism to extend such quasigeostrophic calculations to the more realistic shallow water model.

The properties of the shallow water system are first recalled in Sec. II stressing the conservation laws, the backbone of the statistical mechanics approach. In Sec. III the procedure of Robert and Sommeria [17] is extended to the shallow water system with discussion of simplified cases in Sec. IV. We still assume that the vorticity field creates intricate filamentation but the divergence and water height (surface density) fields are still smooth. These assumptions are justified for flows dominated by vortical motion at moderate Mach numbers, for which the generation of shocks is not effective. This is the case in most cases of atmospheric or oceanic dynamics.

The relaxation toward equilibrium is discussed in Sec. V providing methods for determining the statistical equilibrium. It can be used also as a subgrid-scale modeling for shallow water turbulence, and this is probably the main practical application of such statistical mechanics approaches [27,32]: indeed most cases of interest correspond to forced systems, which never really reach a statistical equilibrium, but should be permanently driven toward such equilibrium by eddy transport. Finally the case of particular domain geometries is discussed in Sec. VI.

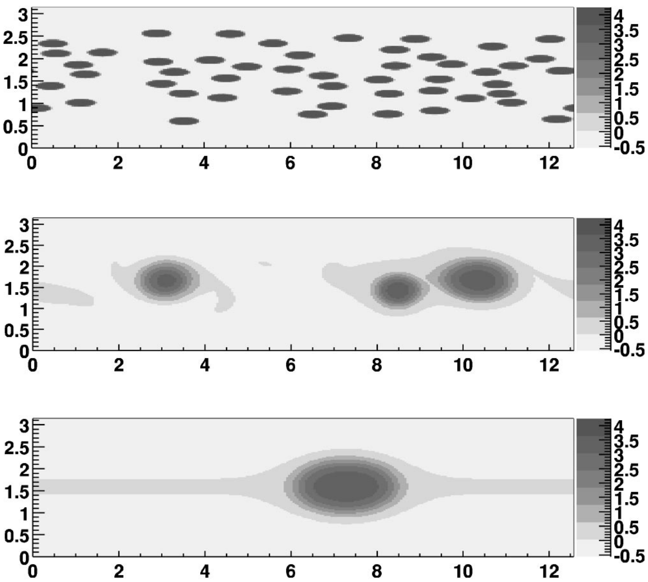

FIG. 1. Relaxation towards statistical equilibrium in a QG model of the great red spot (from Ref. [40]) . Three successive PV fields are represented as gray levels. The initial condition (top) is made of small PV patches. These patches organize into vortices (middle) that eventually merge into a single one (bottom). This sequence is obtained by the QG version of the relaxation equations generalized to shallow water in Sec. V: entropy increases with time while energy is exactly conserved. At equilibrium, the vortex is an oval spot of quasiuniform PV surrounded by strong gradients, corresponding to an annular jet.

II. THE SHALLOW WATER EQUATIONS

Consider a fluid layer with thickness $h(x,y,t)$ submitted to a gravity acceleration g on a rotating planet. We assume that the layer is thin with respect to the characteristic length scale of the horizontal motion. In that case, the velocity field $\mathbf{u}(x,y,t)$ can be assumed two dimensional and the vorticity $\boldsymbol{\omega}=\omega\mathbf{e}_z=\nabla\times\mathbf{u}$ is directed along the vertical axis. We shall assume for simplicity a plane geometry, with rotation vector $\boldsymbol{\Omega}$ directed along the vertical, but extension to a sphere would be straigthforward (we introduce the Coriolis effect but no centrifugal force as the latter is incorporated in the gravity of the planet). The time evolution of these quantities is determined by the shallow water equations [33]

$$\frac{\partial h}{\partial t}+\boldsymbol{\nabla}\cdot(h\mathbf{u})=0, \tag{1}$$

$$\frac{\partial \mathbf{u}}{\partial t}+(\boldsymbol{\omega}+2\boldsymbol{\Omega})\times\mathbf{u}=-\boldsymbol{\nabla}B. \tag{2}$$

Here, the usual advective term $\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u}$ has been expressed using the well-known identity of vector analysis $\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u}=\boldsymbol{\nabla}(\mathbf{u}^2/2)+\boldsymbol{\omega}\times\mathbf{u}$, and the term $\mathbf{u}^2/2$ incorporated in the Bernouilli function

$$B = gh + \frac{\mathbf{u}^2}{2}, \tag{3}$$

together with the pressure term gh. Note that the shallow water system can be viewed as a 2D flow of a compressible gas with density h and equation of state $p = gh^2/2$ (p is the vertically integrated pressure) and our results could be readily generalized to 2D compressible adiabatic flows. We shall often refer to the shallow water system as the "compressible case," by opposition with the "incompressible" case, for which Eq. (1) is replaced by $\nabla \cdot \mathbf{u} = 0$.

One can easily check that the PV

$$q = \frac{\omega + 2\Omega}{h} \tag{4}$$

is conserved for each fluid parcel, i.e.,

$$\frac{dq}{dt} \equiv \frac{\partial q}{\partial t} + \mathbf{u} \cdot \nabla q = 0. \tag{5}$$

Each mass element $hd^2\mathbf{r}$ is conserved during the course of the evolution. Together with Eq. (5), this implies the conservation of the Casimir invariants

$$C_f = \int f(q) h d^2\mathbf{r}, \tag{6}$$

where f is any continuous function of the potential vorticity. In particular, the moments Γ_n of the potential vorticity are conserved

$$\Gamma_n = \int q^n h d^2\mathbf{r}. \tag{7}$$

The moments $n = 0,1,2$ are, respectively, the total mass M, the circulation Γ, and the PV enstrophy Γ_2. The energy

$$E = \int h \frac{\mathbf{u}^2}{2} d^2\mathbf{r} + \frac{1}{2} \int gh^2 d^2\mathbf{r}, \tag{8}$$

involving a kinetic and a potential part, is also a conserved quantity. Note finally that for a multiply connected domain, e.g., the annulus or the channel discussed in Sec. VI, the circulation along each boundary is conserved, in addition to Γ.

It will be convenient in the sequel to use a Helmholtz decomposition of the momentum $h\mathbf{u}$ into a purely rotational and a purely divergent part

$$h\mathbf{u} = -\mathbf{e}_z \times \nabla \psi + \nabla \varphi, \tag{9}$$

where ψ and φ are defined as solutions of the Poisson equations

$$\Delta \psi = -\nabla \times (h\mathbf{u}), \quad \psi = \text{const} \quad \text{on each boundary}, \tag{10}$$

$$\Delta \varphi = -\nabla \cdot (h\mathbf{u}), \quad \partial\varphi/\partial\zeta = 0 \quad \text{on each boundary}. \tag{11}$$

The conditions at the domain boundary (with normal coordinate ζ) are the consequences of the wall impermeability. We here consider a domain with a single (outer) boundary, so we can take $\psi = 0$ at this boundary (as ψ is defined within an arbitrary gauge constant).

For a steady solution, the mass conservation (1) reduces to $\nabla \cdot (h\mathbf{u}) = 0$, so that $\varphi = 0$ and

$$h\mathbf{u} = -\mathbf{e}_z \times \nabla \psi \quad \text{(steady)}. \tag{12}$$

Equation (5) then reduces to $\mathbf{u} \cdot \nabla q = 0$, which implies that q is a function F of the stream function ψ

$$q = F(\psi). \tag{13}$$

Finally, Eq. (2) reduces to

$$(\boldsymbol{\omega} + 2\boldsymbol{\Omega}) \times \mathbf{u} = -\nabla B. \tag{14}$$

Taking the dot product with $\mathbf{u}$, we obtain $\mathbf{u} \cdot \nabla B = 0$ or, equivalently, $B = B(\psi)$. This is known as the Bernouilli theorem. Substituting Eq. (12) in Eq. (14), we obtain a specific relationship between the potential vorticity q and the Bernouilli function B in the form

$$q = -\frac{dB}{d\psi}. \tag{15}$$

Small perturbations to a state of rest, with uniform thickness H, satisfy linearized equations with two branches of solutions. For small scales, these are the usual surface waves on one hand, with purely divergent velocity and propagation speed $c = (gH)^{1/2}$, and steady vortical divergenceless modes on the other hand. In nonlinear regimes, these two modes interact. Vortical motion with scale l and typical vorticity ω fluctuates on time scale ω^{-1}, so it emits waves at wavelength $\lambda \sim c/\omega$. Hence λ/l is the inverse of the Mach number c/u based on the local velocity $u \sim \omega l$. Therefore, we expect that for our considered case of small Mach numbers, velocity divergence and free surface deformation are much smoother than the vorticity field (their wavelength is much larger).

III. THE EQUILIBRIUM STATISTICAL MECHANICS

A. General principles and notations

For slow velocities, the shallow water system reduces to the quasigeostrophic (QG) equations, with $h \simeq cte$, such that Eq. (1) reduces to the incompressibility condition $\nabla \cdot \mathbf{u} = 0$. Then the velocity field remains regular for any time, but it generally develops complex fine-scale vorticity filaments so that a deterministic description of the flow would require a rapidly increasing amount of information as time goes on. The idea of the statistical theory is to give up such a *deterministic* description and refer to a *probabilistic* description. Therefore, the exact knowledge of the "fine-grained," or microscopic potential vorticity field is replaced by the probability density (area fraction) $\rho(\mathbf{r}, \sigma)$ of finding the potential vorticity level σ at position $\mathbf{r}$. We extend here the same definition to the shallow water case. The normalization condition yields at each point

$$\int \rho(\mathbf{r},\sigma)d\sigma = 1. \tag{16}$$

The locally averaged field of potential vorticity is expressed in terms of the probability density in the form

$$\overline{q} = \int \rho(\mathbf{r},\sigma)\sigma d\sigma. \tag{17}$$

The statistical equilibrium will correspond to the probability field $\rho(\mathbf{r},\sigma)$ that maximizes a mixing entropy with the integral constraints brought by the conserved quantities.

In the incompressible case, the velocity field is supposed smooth, as it is obtained from the vorticity field by a spatial integration process, and the integration over the local fluctuations yields $\overline{q}$ (see Ref. [17] for a precise justification). By contrast the inviscid shallow water dynamics generally leads to singularities (shocks), with associated energy dissipation even in the absence of viscosity. This is a source of fundamental mathematical difficulty for the generalization of the equilibrium statistical mechanics initially developed for the Euler equations or QG system. Nevertheless, for the case of small Mach numbers that we consider, shocks occur only after a very long time (due to nonlinear steepening of surface waves), and they involve a weak energy dissipation, since most of the energy remains in the vortical motion. Furthermore, fine-scale vorticity fluctuations behave in the same way as they do in the QG system, and only interact with surface waves and flow divergence at much larger scale, as discussed above. In conclusion, we shall assume that vorticity fluctuates at small scale but we assume that velocity is smooth as well as its divergence and the height h.

A macroscopic state is then defined by its probability field $\rho(\mathbf{r},\sigma)$, its height field $h(\mathbf{r})$, and its flow divergence (both assumed without microscopic fluctuations). The velocity field $\mathbf{u}(\mathbf{r})$ can be deduced by integration from its divergence and vorticity $\overline{\omega} = \overline{q}h - 2\Omega$ [using a Helmholz decomposition for $\mathbf{u}$ analogous to Eqs. (9)–(11)]. The energy (8) depends only on this smooth field, with negligible influence of local fluctuations, as in the incompressible case [17]. The conservation of the Casimir invariant (6) is equivalent to the conservation of the global distribution of potential vorticity (i.e., the total area of each level of potential vorticity weighted by h)

$$\gamma(\sigma) = \int \rho(\mathbf{r},\sigma)h(\mathbf{r})d^2\mathbf{r}. \tag{18}$$

The microscopic moments of potential vorticity can be written

$$\Gamma_n = \int \gamma(\sigma)\sigma^n d\sigma = \int \overline{q^n h}(\mathbf{r})d^2\mathbf{r}, \tag{19}$$

where

$$\overline{q^n} = \int \rho(\mathbf{r},\sigma)\sigma^n d\sigma, \tag{20}$$

and they are conserved during an inviscid evolution. By contrast, the macroscopic moments of potential vorticity $\Gamma_n^{cg} = \int \overline{q}^n h d^2\mathbf{r}$ are *not* conserved for $n \geqslant 2$, as they are partly transferred into fine-grained fluctuations ($\Gamma_2^{cg} = \int \overline{q}^2 h d^2\mathbf{r}$ is the coarse-grained potential enstrophy).

The mixing entropy

$$S = -\int \rho(\mathbf{r},\sigma)\ln \rho(\mathbf{r},\sigma)h(\mathbf{r})d^2\mathbf{r}d\sigma, \tag{21}$$

measures the number of microscopic configurations associated with the same macroscopic field of potential vorticity. The dependence in ρ is the same as for the incompressible case [17], and the factor $h(\mathbf{r})$ is introduced to ensure that entropy is conserved by mere macroscopic displacement of fluid parcels. Indeed, the mass element $h(\mathbf{r})d^2\mathbf{r}$ is conserved by fluid particle displacement instead of the surface element $d^2\mathbf{r}$ in the incompressible case. At equilibrium, the system is expected to be in the most probable (i.e., most mixed) state consistent with the constraints of the Euler equation. The entropy (21) has been justified by rigorous mathematical arguments (in the incompressible case) but other forms have been recently proposed [20]. Therefore, we shall consider a general form of entropy

$$S = \int s(\rho(\mathbf{r},\sigma))h(\mathbf{r})d^2\mathbf{r}d\sigma, \tag{22}$$

and find that Eq. (21) is the only expression leading to an entropy extremum.

B. First order variations

According to the previous discussion, the most probable macroscopic state is obtained by maximizing the entropy (22) with fixed energy (8), global vorticity distribution (18), and local normalization (16). This problem is treated by introducing Lagrange multipliers, so that the first variations satisfy

$$\delta S - \beta\delta E - \int \alpha(\sigma)\delta\gamma(\sigma)d\sigma - \int \zeta(\mathbf{r})\delta\left(\int \rho(\mathbf{r},\sigma)d\sigma\right)hd^2\mathbf{r} = 0. \tag{23}$$

The Lagrange multipliers are, respectively, the "inverse temperature" β, the "chemical potential" $\alpha(\sigma)$ of PV species σ, and $\zeta(\mathbf{r})$.

We shall take $h(\mathbf{r})$, $\rho(\mathbf{r},\sigma)h(\mathbf{r})$, and $\nabla\cdot\mathbf{u}(\mathbf{r})$ as independent variables characterizing the macroscopic state. Then, it is easy to establish, by differentiating, respectively, Eqs. (22), (18), and (16), that

$$\delta S = \int [-\rho s'(\rho) + s(\rho)]\delta h d^2\mathbf{r}d\sigma + \int s'(\rho)\delta(\rho h)d^2\mathbf{r}d\sigma, \tag{24}$$

$$\delta\gamma(\sigma) = \int \delta(\rho h)d^2\mathbf{r}, \tag{25}$$

$$h\delta\left(\int \rho(\mathbf{r},\sigma)d\sigma\right)=\int \delta(\rho h)d\sigma-\int \rho\delta h d\sigma. \quad (26)$$

The variations of energy are conveniently expressed in terms of the Bernouilli function B as

$$\delta E=\int B\delta h d^2\mathbf{r}+\int h\mathbf{u}\cdot\delta\mathbf{u}d^2\mathbf{r}. \quad (27)$$

Then, using the Helmholtz decomposition (9) for the momentum $h\mathbf{u}$, the second integral can be rewritten

$$\int h\mathbf{u}\cdot\delta\mathbf{u}d^2\mathbf{r}=-\int(\nabla\psi\times\delta\mathbf{u})\cdot\mathbf{e}_z d^2\mathbf{r}+\int\nabla\varphi\cdot\delta\mathbf{u}d^2\mathbf{r}. \quad (28)$$

Integrating by parts with the identities $\nabla\times(\psi\delta\mathbf{u})=\psi\nabla\times\delta\mathbf{u}+\nabla\psi\times\delta\mathbf{u}$ and $\nabla\cdot(\varphi\delta\mathbf{u})=\varphi\nabla\cdot(\delta\mathbf{u})+\nabla\varphi\cdot\delta\mathbf{u}$, and using the boundary conditions for ψ and φ , we obtain

$$\int h\mathbf{u}\cdot\delta\mathbf{u}d^2\mathbf{r}=\int\psi\delta\overline{\omega}d^2\mathbf{r}-\int\varphi\nabla\cdot(\delta\mathbf{u})d^2\mathbf{r}. \quad (29)$$

Using Eqs. (16), (4), and (17), we have finally

$$\delta E=\int B\rho\delta h d^2\mathbf{r}d\sigma+\int\psi\sigma\delta(\rho h)d^2\mathbf{r}d\sigma-\int\varphi\delta(\nabla\cdot\mathbf{u})d^2\mathbf{r}. \quad (30)$$

The variation (23) vanishes for any small changes of the variables only if the coefficient of each independent variable vanishes

$$\delta(\rho h)\quad \text{if}\quad s'(\rho)=\beta\sigma\psi+\alpha(\sigma)+\zeta(\mathbf{r}), \quad (31)$$

$$\delta h\quad \text{if}\quad -s'(\rho)+\frac{s(\rho)}{\rho}=\beta B-\zeta(\mathbf{r}), \quad (32)$$

$$\delta(\nabla\cdot\mathbf{u})\quad \text{if}\quad \varphi=0. \quad (33)$$

Note that the right-hand side of Eq. (32) is independent of σ. This implies that the term on the left-hand side must be a constant (that we can take equal to 1 without loss of generality): $-s'(\rho)+s(\rho)/\rho=1$. This equation is easily integrated in $s(\rho)=A\rho-\rho\ln\rho$ where A is an integration constant. When substituted in Eq. (22), using Eq. (19), this yields

$$S=-\int\rho(\mathbf{r},\sigma)\ln\rho(\mathbf{r},\sigma)h(\mathbf{r})d^2\mathbf{r}d\sigma+AM, \quad (34)$$

which is just the entropy (21) up to an additive constant term AM (which we can take equal to zero without loss of generality). Therefore, the entropy (21) is the only functional of the form (22) for which the maximization problem has a solution. This result is astounding because it is obtained without any explicit reference to thermodynamical arguments.

C. The Gibbs states

Substituting explicitly $s(\rho)=-\rho\ln\rho$ in Eq. (31), the optimal probability density can be expressed as

$$\rho(\mathbf{r},\sigma)=\frac{1}{Z(\psi)}g(\sigma)e^{-\beta\sigma\psi}, \quad (35)$$

where $Z(\psi)\equiv e^{\zeta(\mathbf{r})+1}$ and $g(\sigma)\equiv e^{-\alpha(\sigma)}$. The normalization condition (16) leads to a value of the partition function Z of the form

$$Z=\int g(\sigma)e^{-\beta\sigma\psi}d\sigma, \quad (36)$$

and the locally averaged potential vorticity (17) is expressed as a function of ψ according to

$$\overline{q}=\frac{\int g(\sigma)\sigma e^{-\beta\sigma\psi}d\sigma}{\int g(\sigma)e^{-\beta\sigma\psi}d\sigma}=F(\psi). \quad (37)$$

This expression can be rewritten

$$\overline{q}=-\frac{1}{\beta}\frac{d\ln Z}{d\psi}, \quad (38)$$

which is the same functional relation as in the case of 2D incompressible Euler flows [17] or quasigeostrophic equations.

Differentiating Eq. (37) with respect to ψ, we check that the variance of the potential vorticity can be written

$$q_2\equiv\overline{q^2}-\overline{q}^2=-\frac{1}{\beta}F'(\psi), \quad (39)$$

or, alternatively [see Eq. (38)]

$$q_2=\frac{1}{\beta^2}\frac{d^2\ln Z}{d\psi^2}. \quad (40)$$

Therefore, the slope of the function $\overline{q}=F(\psi)$ is directly related to the variance of the vorticity distribution. A similar result [41] links the successive moments to the successive higher order derivatives of F. Relation (39) has the same form and origin as the "fluctuation-dissipation" theorem in statistical field theory, where $d\overline{q}/d\psi$ is interpreted as a susceptibility [42]. Since $q_2>0$, we find that the function $\overline{q}=F(\psi)$ is monotonic; it is decreasing for $\beta>0$ and increasing for $\beta<0$ (it is constant for $\beta=0$). Another proof of this result is given in Ref. [17].

Substituting explicitly $-s'(\rho)+s(\rho)/\rho=1$ in Eq. (32), we have

$$B=\frac{1}{\beta}\ln Z. \quad (41)$$

This relation shows that the Bernouilli function plays the role of a free energy in the statistical theory. We note that both B

and $\overline{q}$ are functions of ψ, while $\varphi=0$ from Eq. (33), as it should for steady flows. Furthermore, taking the derivative of Eq. (41) with respect to ψ and using Eq. (38), we check that the relation $\overline{q}=-dB/d\psi$ required for a steady solution of the shallow water equation is satisfied. Therefore, the flow selected by our purely statistical procedure is preserved by the flow evolution, so the statistical theory is indeed consistent with the dynamics.

In order to get an explicit prediction of the self-organized state for any given initial condition, the Eqs. (41) and $\overline{q}=-dB/d\psi$ must be solved. Using $\overline{q}=(\overline{\omega}+2\Omega)/h$, with $\overline{\omega}=-\nabla\times(e_z\times\nabla\psi/h)$, and the definition (3) of B, these two equations can be written as two coupled partial differential equations for ψ and h

$$-\frac{\Delta\psi}{h^2}+\frac{2\Omega}{h}+\frac{1}{h^3}\nabla\psi\nabla h=-\frac{dB}{d\psi}, \tag{42}$$

$$\frac{(\nabla\psi)^2}{2h^2}+gh=B(\psi). \tag{43}$$

The function $B(\psi)=(1/\beta)\ln Z(\psi)$ depends on the inverse temperature β and the function $g(\sigma)$, and these quantities are implicitly defined by the integral constraints on energy (8) and global PV probability distribution (18), $\gamma(\sigma)=\int[g(\sigma)/Z(\psi)]e^{-\beta\sigma\psi}h(\mathbf{r})d^2\mathbf{r}$. This problem has several solutions in general (this is already the case for 2D incompressible flows, see Refs. [43,44]), but most of them are not true entropy maxima: the first variations (23) are canceled but the sign of the second variations should be checked. A good method to select the entropy maxima is to use a relaxation process that, starting from the initial condition, increases entropy while keeping constant the integral constraints. In the incompressible case, such a process has been performed by a continuous time relaxation equation [45], or directly by a discrete time relaxation algorithm [46]. We shall extend the continuous time relaxation to the shallow water system in Sec. V, but we first discuss some simple particular cases.

IV. PROPERTIES OF THE GIBBS STATES IN SOME PARTICULAR CASES

A. Particular $\overline{q}-\psi$ relationships

The Gibbs states are characterized by the relation (37) between $\overline{q}$ and ψ, which is always monotonic, as shown in the preceding section. For numerical calculations of the Gibbs state, we need to discretize the PV levels, replacing the integrals over PV levels by sums. Calculations in the incompressible case [19,20,27] indicate that the result already converges with a few PV levels. Keeping only two levels, $q=\sigma_0$ and $q=\sigma_1$, it is convenient to simplify the discussion and is often representative of more general cases. Then, the probability distribution ρ just depends on a single probability p_1 of finding the level σ_1 (for instance), with the probability $1-p_1$ of finding the complementary level σ_0. This probability p_1 is directly related to the PV average by $\overline{q}=p_1\sigma_1+(1-p_1)\sigma_0$, or reversely $p_1=(\overline{q}-\sigma_0)/(\sigma_1-\sigma_0)$. Then, $g(\sigma)$ is the sum of two Dirac function terms, $g(\sigma)=g_0\delta(\sigma-\sigma_0)+g_1\delta(\sigma-\sigma_1)$, so the expression (36) becomes $Z=g_0\exp(-\beta\sigma_0\psi)+g_1\exp(-\beta\sigma_1\psi)$. The corresponding expression (37) for $\overline{q}$ reduces to

$$\overline{q}=\sigma_0+\frac{\sigma_1-\sigma_0}{1+\lambda\exp\beta(\sigma_1-\sigma_0)\psi}. \tag{44}$$

This relation corresponds to the Fermi-Dirac statistics. The two unknown parameters $\lambda\equiv g_0/g_1$ and β are indirectly determined by the conserved quantities. The associated Bernouilli function (41) becomes

$$B=\frac{1}{\beta}\ln g_1-\sigma_0\psi+\frac{1}{\beta}\ln\{\lambda+e^{\beta(\sigma_0-\sigma_1)\psi}\}. \tag{45}$$

The problem is also greatly simplified in the alternative case for which $g(\sigma)$ is a Gaussian

$$g(\sigma)=g_0e^{[-(\sigma-\sigma_*)^2]/2\sigma_2}. \tag{46}$$

Then, the local probability distribution (35) is also a Gaussian, and the corresponding Bernouilli function (41) is

$$B=\frac{1}{\beta}\ln[g_0(2\pi\sigma_2)^{1/2}]+\frac{1}{2}\sigma_2\beta\psi^2-\sigma_*\psi \tag{47}$$

corresponding to a linear relationship

$$\overline{q}=-\beta\sigma_2\psi+\sigma_*. \tag{48}$$

According to Eq. (39) the variance of the potential vorticity is then uniform, with value $q_2=\sigma_2$ (more generally, all the even moments of the Gaussian are related to σ_2 by $\overline{(q-\overline{q})^{2n}}=(2n-1)!!\sigma_2^n$ and the odd moments cancel out).

This Gaussian local probability distribution is obtained by maximizing the entropy (21), reducing the constraints of the global distribution $\gamma(\sigma)$ to its first moments $\Gamma_0\equiv M$, Γ_1 and Γ_2. This will be the true Gibbs state for a particular initial distribution $\gamma(\sigma)$ with higher order moments equal to the global moments of this simplified Gibbs state. A linear relationship between $\overline{q}$ and ψ can also be obtained for any distribution $\gamma(\sigma)$ in the limit of strong mixing where $\beta\sigma\psi\ll1$, so that Eq. (37) can be linearized, as discussed by Chavanis and Sommeria [43]. In both cases, the result is equivalent to the state of minimum (coarse-grained) potential enstrophy Γ_2^{cg} with the constraints of given E, M, and Γ.

B. Unidirectional solutions

We consider here unidirectional solutions such that $\mathbf{u}=u(y)\mathbf{e}_x$. The equilibrium relation $\overline{q}=-dB/d\psi$ then yields, multiplying each member by $hu=d\psi/dy$ and using the expressions (3) and (4),

$$g\frac{dh}{dy}+\frac{2\Omega}{h}\frac{d\psi}{dy}=0. \tag{49}$$

This condition of geostrophic balance can be readily integrated in

$$h^2+\frac{4\Omega}{g}\psi=H^2, \tag{50}$$

where H is an integration constant. Therefore, the height h is a decreasing function of ψ.

A second relation is provided by the expression (43) of the Bernouilli function. It is convenient to scale the y coordinate by the Rossby radius of deformation, writing $y=L_R\xi$ with

$$L_R=\frac{\sqrt{gH}}{2\Omega}. \tag{51}$$

Then, Eq. (43) yields a first order ordinary differential equation for h,

$$\frac{1}{2}\left(\frac{dh}{d\xi}\right)^2=H[B(h)/g-h]\equiv U(h). \tag{52}$$

The Gibbs state expression (41) of $B(\psi)$ can be expressed as a function $B(h)$ of h using Eq. (50). Note that the solution must be viewed generally in a x-wise translating frame of reference, as discussed in Sec. VI, where the additional conservation law for momentum is included.

Equation (52) directly generalizes the QG equation discussed in Ref. [28]. Free jet solutions, separating two regions of uniform height $h=h_{\pm 1}$, are possible, provided that the function $U(h)$ defined in Eq. (52) vanishes for $h=h_{\pm 1}$ as well as its derivative $U'(h)$. This is a good representation for the annular jet structure of the great red spot or the intense eastward jet in jupiter's northern hemisphere. Such a free jet solution is possible, for instance, in the two PV level case (45), with appropriate relationship between the Lagrange parameters characterizing a phase equilibrium between the two uniform regions [28,47].

Boundary currents, for instance in the half space $\xi>\xi_0$ (with any origin ξ_0), are also typical solutions of Eq. (52). We shall take, by convention, $\psi\rightarrow 0$ as $\xi\rightarrow+\infty$. Then, $U(h)$ and its derivative $U'(h)$ must vanish for the asymptotic height $h=H$, reached at large distance ξ [see Eq. (50)]. For $\psi\rightarrow 0$, the Gibbs function $q(\psi)$ can be linearized to first order in ψ. Then, due to the relation $q=-dB/d\psi$, $B(\psi)$ is quadratic in ψ, and using Eq. (50), we have $U(h)$ of the form $U(h)=H^2[(\lambda/2)(h/H)^4-b(h/H)^2-(h/H)+c]$, where $\lambda=-(L_R^2H^2/4)(dq/d\psi)_{(\psi=0)}$. According to Eq. (39), λ has the same sign as the inverse temperature β. The conditions $U(H)=U'(H)=0$ imply the relationships $b=\lambda-1/2$ and $c=(1+\lambda)/2$ between the coefficients. Then, Eq. (52) can be written in the form

$$\left(\frac{d\phi}{d\xi}\right)^2=(1-\phi)^2(1+\lambda(1+\phi)^2), \quad \text{with} \quad \phi\equiv h/H, \tag{53}$$

in this case of a linearized function $q(\psi)$. Writing $\phi=1+\theta$ and expanding Eq. (53) to lowest order in θ, we find the asymptotic behavior $\theta\sim\exp(-\sqrt{1+4\lambda}\,\xi)$ as $\xi\rightarrow+\infty$ (this imposes $\lambda\geqslant-1/4$). Returning to the original variables, we find that the boundary current extends on a typical length $\sim L_R/\sqrt{1+4\lambda}$. If $\beta>0$, this length is smaller than the Rossby radius and if $\beta<0$, it is larger.

In the case of a linear relationship between q and ψ, Eq. (53) is valid in the whole space $\xi>\xi_0$. We have found particular analytical solutions that are representative of the general behavior for any λ. For $\lambda=0$ (i.e., $\beta=0$ or $\overline{q}=\text{const}$), we find

$$\phi=1-e^{-\xi}. \tag{54}$$

For $\lambda\gg 1$ (i.e., β large), we find

$$\phi=\tanh(\sqrt{\lambda}\,\xi), \tag{55}$$

for an appropriate origin of ξ such that the reduced thickness ϕ remains positive. We can check explicitly on these examples that $\theta=\phi-1$ is attenuated exponentially on a typical length $1/\sqrt{1+4\lambda}$. For $\lambda=-1/4$, the decay length diverges and we have a power law behavior

$$\phi=\frac{(\xi+\sqrt{3})^2-3}{(\xi+\sqrt{3})^2+1} \tag{56}$$

leading to $\theta\sim-4/\xi^2$ as $\xi\rightarrow+\infty$. This ξ^{-2} behavior is valid for any Bernouilli function when $\lambda=-1/4$.

These solutions describe boundary jets. We can show that there are no jet solutions separating two domains with uniform PV for a linear $q-\psi$ relationship, unlike in the more general case, for instance with two PV levels. A similar conclusion was already reached in the QG case [28]. For the above boundary jet solutions, h increases with distance to the boundary, such that the pressure gradient balances a Coriolis force directed away from the boundary. Another possibility, for a boundary current in the opposite direction, is a confined solution with h reaching the value zero at a finite wall distance. The absolute vorticity $2\Omega-du/dy$ must vanish at this position to ensure finiteness of the PV $(2\Omega-du/dy)/h$. The free surface slope remains finite, as well as the velocity u, and both are related by the geostrophic balance $dh/dy=-2\Omega u$. Such finite support solutions do not exist in the QG case. Note finally that all the solutions discussed here cancel the first variations of entropy (23) but they are not necessarily entropy *maxima*. This has to be checked, for instance, by the relaxation equations discussed in Sec. V.

C. Axisymmetric solutions

For axisymmetric solutions, $\mathbf{u}=u_\theta(r)\mathbf{e}_\theta$ (where (r,θ) are polar coordinates) and $hu_\theta=-d\psi/dr$. Then, Eq. (49) is replaced by the cyclostrophic balance

$$gh\frac{dh}{dr}=\frac{1}{hr}\left(\frac{d\psi}{dr}\right)^2-2\Omega\frac{d\psi}{dr}. \tag{57}$$

When $hu_\theta=-d\psi/dr\geqslant 0$ (cyclone), h is an increasing function of r, so the vortex core is a trough. In the opposite case

$u_\theta \leq 0$ (anticyclone), the vortex core is a bump in geostrophic regimes. However, for large velocities (Rossby number larger than unity), the term u_θ^2/r dominates, leading always to a trough.

Combining this relation (57) with the expression (43), one gets a couple of first order ordinary differential equations in the variables ψ and h. As in the unidirectional case, the solution depends on two constants of integration and the Lagrange parameters, which are for example, g_1, β, and λ in the case with two PV levels. This solution must be viewed in general in a rotating frame of reference, due to the additional conservation of angular momentum, as discussed in Sec. VI.

V. RELAXATION EQUATIONS

A. The maximum entropy production principle

Relaxation methods are convenient to compute the statistical equilibrium resulting from any initial condition. The aim is to increase entropy in successive steps while keeping constant all the conserved quantities. Turkington and Whitaker [46] have implemented a relaxation method to calculate the Gibbs states obtained with the Euler equations. Robert and Sommeria [45] had previously proposed relaxation equations in the form of a parametrization of subgrid-scale eddies that drives the system toward statistical equilibrium by a continuous time evolution. Such relaxation equations can be used both as a realistic coarse resolution model of the turbulent evolution, and as a method of determination of the statistical equilibrium resulting from this evolution (see Ref. [5] for a short review). We here generalize this approach to the shallow water system.

We first decompose the vorticity ω and velocity $\mathbf{u}$ into a mean and fluctuating part, namely, $\omega=\overline{\omega}+\tilde{\omega}$, $\mathbf{u}=\overline{\mathbf{u}}+\tilde{\mathbf{u}}$, keeping h smooth. Taking the local average of the shallow water equations (1) and (2), we get

$$\frac{\partial h}{\partial t}+\nabla\cdot(h\overline{\mathbf{u}})=0, \tag{58}$$

$$\frac{\partial\overline{\mathbf{u}}}{\partial t}+(\overline{\boldsymbol{\omega}}+2\boldsymbol{\Omega})\times\overline{\mathbf{u}}=-\nabla\overline{B}-\mathbf{e}_z\times\mathbf{J}_\omega, \tag{59}$$

$$\overline{B}=gh+\frac{\overline{\mathbf{u}}^2}{2}, \tag{60}$$

where the current $\mathbf{J}_\omega=\overline{\tilde{\omega}\tilde{\mathbf{u}}}$ represents the correlations of the fine-grained fluctuations. Although we have neglected the fluctuation energy $\tilde{\mathbf{u}}^2$ in front of $\overline{\mathbf{u}}^2$ (as well as fluctuations of h), we keep the correlations $\mathbf{J}_\omega=\overline{\tilde{\omega}\tilde{\mathbf{u}}}$, which represent the PV transport by subgrid-scale eddies. This assumption is justified since, denoting by ϵ the typical scale of vorticity fluctuations, we have $\tilde{\mathbf{u}}^2\sim\epsilon^2\overline{\omega}^2$ and $\overline{\tilde{\omega}\tilde{\mathbf{u}}}\sim\epsilon\overline{\omega}^2\gg\tilde{\mathbf{u}}^2$ (while $\tilde{\omega}\sim\overline{\omega}$).

We deduce an equation for the evolution of the potential vorticity (4), taking the curl of Eq. (59) and using Eq. (58)

$$\frac{\partial}{\partial t}(h\overline{q})+\nabla\cdot(h\overline{q}\,\overline{\mathbf{u}})=-\nabla\cdot\mathbf{J}_\omega. \tag{61}$$

This equation can be viewed as a local conservation law for the circulation $\Gamma=\int\overline{q}hd^2\mathbf{r}$. We shall determine the unknown current $\mathbf{J}_\omega$ by the thermodynamic principle of maximum entropy production (MEP) [45]. For that purpose, we need to consider not only the locally averaged PV field $\overline{q}$, but also the whole probability distribution $\rho(\mathbf{r},\sigma,t)$ now evolving with time t. The conservation of the global vorticity distribution $\gamma(\sigma)=\int\rho hd^2\mathbf{r}$ can be written in the local form

$$\frac{\partial}{\partial t}(h\rho)+\nabla\cdot(h\rho\overline{\mathbf{u}})=-\nabla\cdot\mathbf{J}, \tag{62}$$

where $\mathbf{J}(\mathbf{r},\sigma,t)$ is the (unknown) current associated with the level σ of potential vorticity. Integrating Eq. (62) over all the PV levels σ, using Eq. (16), and comparing with Eq. (58), we find the constraint

$$\int\mathbf{J}(\mathbf{r},\sigma,t)d\sigma=\mathbf{0}. \tag{63}$$

Multiplying Eq. (62) by σ, integrating over all the PV levels, using Eq. (17), and comparing with Eq. (61), we get

$$\int\mathbf{J}(\mathbf{r},\sigma,t)\sigma d\sigma=\mathbf{J}_\omega. \tag{64}$$

We can express the time variation of energy $\dot{E}\equiv dE/dt$ in terms of $\mathbf{J}$, using Eqs. (8) and (59), leading to the energy conservation constraint

$$\dot{E}=\int\mathbf{J}\sigma h\overline{\mathbf{u}}_\perp d^2\mathbf{r}d\sigma=0, \tag{65}$$

where $\overline{\mathbf{u}}_\perp\equiv\mathbf{e}_z\times\overline{\mathbf{u}}$. Using Eqs. (21) and (62), we similarly express the rate of entropy production as

$$\dot{S}=-\int\mathbf{J}\cdot\nabla(\ln\rho)d^2\mathbf{r}d\sigma. \tag{66}$$

The MEP principle consists in choosing the current $\mathbf{J}$ that maximizes the rate of entropy production $\dot{S}$ respecting the constraints $\dot{E}=0$, Eq. (63) and $\int(J^2/2\rho)d\sigma\leq C(\mathbf{r},t)$. The last constraint expresses a bound (unknown) on the value of the diffusion current. Convexity arguments justify that this bound is always reached so that the inequality can be replaced by an equality. The corresponding condition on first variations can be written at each time t

$$\delta\dot{S}-\beta(t)\delta\dot{E}-\int\zeta(\mathbf{r},t)\delta\left(\int\mathbf{J}d\sigma\right)d^2\mathbf{r}$$
$$-\int D^{-1}(\mathbf{r},t)\delta\left(\int\frac{J^2}{2\rho}d\sigma\right)d^2\mathbf{r}=0, \tag{67}$$

and leads to a current of the form

$$\mathbf{J}=-D(\mathbf{r},t)[\nabla\rho+\beta(t)\sigma\rho h\overline{\mathbf{u}}_{\perp}-\boldsymbol{\zeta}(\mathbf{r},t)\rho]. \tag{68}$$

The Lagrange multiplier $\boldsymbol{\zeta}(\mathbf{r},t)$ is determined by the constraint (63), which leads to

$$\mathbf{J}=-D(\mathbf{r},t)[\nabla\rho+\beta(t)\rho(\sigma-\overline{q})h\overline{\mathbf{u}}_{\perp}]. \tag{69}$$

This optimal current is similar to the one obtained in ordinary incompressible 2D turbulence except that the term $h\overline{\mathbf{u}}_{\perp}$ now replaces $\nabla\psi$. The impermeability condition imposes that the normal component of the velocity and of the current vanishes at the wall. We, therefore, have the boundary conditions

$$\mathbf{n}\cdot\overline{\mathbf{u}}=0 \quad \text{(on each boundary)}, \tag{70}$$

$$\mathbf{n}\cdot\nabla\rho=-\beta(t)\rho(\sigma-\overline{q})h\mathbf{n}\cdot\overline{\mathbf{u}}_{\perp} \quad \text{(on each boundary)}, \tag{71}$$

where $\mathbf{n}$ is a unit vector normal to the wall.

The diffusion coefficient D is not determined by the MEP as it depends on the unknown bound C on the current. It can be determined by more systematic procedures inspired from kinetic theories of plasma physics and stellar dynamics as in Refs. [4,48–50] for the Euler equations. In that context, the diffusion coefficient is equal to the variance of the velocity fluctuations multiplied by a correlation time scale. The variance of the velocity fluctuations can, in turn, be expressed in terms of the vorticity distribution by using the Biot and Savart formula. The precise form of the diffusion coefficient is important in order to take into account kinetic confinement and incomplete relaxation [51]. However, for the purpose of reaching the Gibbs state, the diffusion coefficient can simply be chosen arbitrarily. We shall show below that D must nevertheless be positive so as to ensure entropy increase.

The conservation of energy (65) at any time determines the evolution of the Lagrange multiplier $\beta(t)$ according to

$$\beta(t)=-\frac{\int D\nabla\overline{q}h\overline{\mathbf{u}}_{\perp}d^2\mathbf{r}}{\int D(\overline{q^2}-\overline{q}^2)(h\overline{\mathbf{u}}_{\perp})^2d^2\mathbf{r}}. \tag{72}$$

We can now provide an explicit form for the vorticity current $\mathbf{J}_{\omega}$ and introduce it back in the shallow water equation (59). Indeed, using Eqs. (69) and (17), we find

$$\mathbf{J}_{\omega}=-D[\nabla\overline{q}+\beta(t)(\overline{q^2}-\overline{q}^2)h\overline{\mathbf{u}}_{\perp}]. \tag{73}$$

Substituting Eq. (73) in Eq. (59), we obtain

$$\frac{\partial\overline{\mathbf{u}}}{\partial t}+(\overline{\boldsymbol{\omega}}+2\boldsymbol{\Omega})\times\overline{\mathbf{u}}=-\nabla\overline{B}+D[\mathbf{e}_z\times\nabla\overline{q}-\beta(t)(\overline{q^2}-\overline{q}^2)h\overline{\mathbf{u}}]. \tag{74}$$

Since $\beta(t)\leqslant 0$ in relevant situations, the last term in Eq. (74) represents a *forcing* proportional to $\overline{\mathbf{u}}$ that compensates the diffusion caused by the term $\mathbf{e}_z\times\nabla\overline{q}\sim\Delta\overline{\mathbf{u}}$. This additional term depends on the local PV variance $\overline{q^2}-\overline{q}^2$, related to the probability distribution ρ, and we need to keep track of it by solving the probability transport equation (62) in addition to the modified shallow water system (58) and (74). This set of equations increases the entropy (at an optimal rate) while preserving all the conservation laws of the initial inviscid shallow water system. We now check that the steady solutions reached by these relaxation equations are indeed the Gibbs states.

B. Relaxation towards statistical equilibrium

The entropy production (66) can be written

$$\dot{S}=-\int\frac{\mathbf{J}}{\rho}[\nabla\rho+\beta\rho(\sigma-\overline{q})h\mathbf{u}_{\perp}]d^2\mathbf{r}d\sigma$$
$$+\beta\int\mathbf{J}(\sigma-\overline{q})h\mathbf{u}_{\perp}d^2\mathbf{r}d\sigma. \tag{75}$$

Using Eqs. (63) and (65), the second integral is seen to cancel out. Substituting Eq. (69) in the first integral, we find

$$\dot{S}=\int\frac{J^2}{D\rho}d^2\mathbf{r}d\sigma, \tag{76}$$

which is positive provided that $D\geqslant 0$. A stationary solution $\dot{S}=0$ is such that $\mathbf{J}=\mathbf{0}$ yielding, together with Eq. (9),

$$\nabla(\ln\rho)+\beta(\sigma-\overline{q})\nabla\psi=\mathbf{0}. \tag{77}$$

For any reference PV level σ_0, it writes

$$\nabla(\ln\rho_0)+\beta(\sigma_0-\overline{q})\nabla\psi=\mathbf{0}. \tag{78}$$

Subtracting Eqs. (77) and (78), we obtain $\nabla\ln(\rho/\rho_0)+\beta(\sigma-\sigma_0)\nabla\psi=\mathbf{0}$, which is immediately integrated into

$$\rho(\mathbf{r},\sigma)=\frac{1}{Z(\mathbf{r})}g(\sigma)e^{-\beta\sigma\psi}, \tag{79}$$

where $Z^{-1}(\mathbf{r})\equiv\rho_0(\mathbf{r})e^{\beta\sigma_0\psi(\mathbf{r})}$ and $g(\sigma)\equiv e^{A(\sigma)}$, $A(\sigma)$ is a constant of integration. Therefore, entropy increases until the Gibbs state (35) is reached, with $\beta=\lim_{t\to\infty}\beta(t)$. Furthermore, we can show (Chavanis, in preparation) that a stationary solution of these relaxation equations is linearly stable if, and only if, it is an entropy *maximum*. Therefore, this numerical algorithm selects the maxima (and not the minima or the saddle points) among all critical points of entropy. When several entropy maxima subsist for the same values of the constraints, the choice of equilibrium depends on a complicated notion of "basin of attraction" and not simply whether the solution is a local or a global entropy maximum. Similar results are obtained for a simple model of gravitational dynamics [52].

C. Simplified cases

In the case of two PV levels σ_0 and σ_1, the transport equation (62) for the probability p_1 is equivalent to the transport equation for $\overline{q}$ [since $\overline{q}=\sigma_0+p_1(\sigma_1-\sigma_0)$], which is already obtained from the curl of the shallow water equation (74). Therefore, the relaxation equations reduce to the modified shallow water system

$$\frac{\partial h}{\partial t}+\nabla\cdot(h\mathbf{u})=0, \tag{80}$$

$$\frac{\partial \mathbf{u}}{\partial t}+\overline{q}h\mathbf{e}_z\times\mathbf{u}=-\nabla\left(\frac{\mathbf{u}^2}{2}+gh\right)+D[\mathbf{e}_z\times\nabla\overline{q}-\beta(t)q_2h\mathbf{u}], \tag{81}$$

$$\overline{q}=\frac{(\nabla\times\mathbf{u})\cdot\mathbf{e}_z+2\Omega}{h},\quad q_2=(\overline{q}-\sigma_0)(\sigma_1-\overline{q}), \tag{82}$$

$$\beta(t)=-\frac{\int Dh\mathbf{u}_\perp\nabla\overline{q}d^2\mathbf{r}}{\int Dq_2\mathbf{u}_\perp^2h^2d^2\mathbf{r}}, \tag{83}$$

$$\mathbf{n}\cdot\nabla\overline{q}=-\beta(t)q_2h\mathbf{n}\cdot\mathbf{u}_\perp\quad(\text{on each boundary}), \tag{84}$$

$$\mathbf{n}\cdot\mathbf{u}=0\quad(\text{on each boundary}), \tag{85}$$

where we have omitted the overbar on $\mathbf{u}$, and the expression (82) of $q_2=\overline{q^2}-\overline{q}^2$ is easily obtained for a probability distribution with two values σ_0 and σ_1. The numerical implementation of this system will lead to the two PV level Gibbs state.

Stating $q_2=\text{const}$ instead of the expression (82) yields the Gaussian Gibbs state with linear relationship between $\overline{q}$ and ψ. Then, the coefficient $q_2\beta$ used in Eq. (81) is directly obtained from Eq. (83). This is sufficient for the purpose of finding the statistical equilibrium, but more refined relaxation models can be used as in the context of QG models [27,53].

D. The incompressible limit

The case of ordinary 2D incompressible turbulence is recovered in the limit $h\to H$, $q\to\omega$ and $\mathbf{u}=-\mathbf{e}_z\times\nabla\psi$. The relaxation equation (74) then becomes

$$\frac{\partial\overline{\mathbf{u}}}{\partial t}+(\overline{\mathbf{u}}\cdot\nabla)\overline{\mathbf{u}}=-\frac{1}{\rho}\nabla p+D[\Delta\overline{\mathbf{u}}-\beta(t)\omega_2\overline{\mathbf{u}}], \tag{86}$$

where we have used the well-known identity of vector analysis $\Delta\overline{\mathbf{u}}=\nabla(\nabla\cdot\overline{\mathbf{u}})-\nabla\times(\nabla\times\overline{\mathbf{u}})$, which reduces for a two-dimensional incompressible flow to $\Delta\overline{\mathbf{u}}=\mathbf{e}_z\times\nabla\overline{\omega}$. Equation (86) is valid even if D is space dependant unlike with a usual viscosity term. In previous publications this equation was given only in its vorticity form

$$\frac{\partial\overline{\omega}}{\partial t}+\nabla(\overline{\omega\mathbf{u}})=\nabla[D\{\nabla\overline{\omega}+\beta(t)\omega_2\nabla\psi\}], \tag{87}$$

and the equivalence with Eq. (86) is not obvious at first sight when D is space dependent. At equilibrium, we have from Eq. (86) the identity

$$\Delta\overline{\mathbf{u}}=\beta\omega_2\overline{\mathbf{u}}, \tag{88}$$

which can be deduced directly from the Gibbs state. Indeed, for a stationary solution $\overline{\omega}=F(\psi)$, the previous identity $\Delta\overline{\mathbf{u}}=\mathbf{e}_z\times\nabla\overline{\omega}$ becomes $\Delta\overline{\mathbf{u}}=-F'(\psi)\overline{\mathbf{u}}$ that is equivalent to Eq. (88) for a Gibbs state thanks to Eq. (39).

We now account for a small deformation of the fluid layer but assume that the elevation with respect to the average thickness H is weak, so that

$$h=H(1+\eta)\quad\text{with}\quad\eta\ll1. \tag{89}$$

To the first order, the flow is incompressible and Eq. (1) reduces to $\nabla\cdot\mathbf{u}=0$, or equivalently $\mathbf{u}=-\mathbf{e}_z\times\nabla\psi$ [there is a factor H with the previous definition (9)]. In the quasigeostrophic limit of small Rossby numbers $\omega\ll\Omega$, the momentum equation (2) reduces at zero order to the geostrophic balance

$$\mathbf{u}=\frac{gH}{2\Omega}\mathbf{e}_z\times\nabla\eta\quad\text{or}\quad\psi=-\frac{gH^2}{2\Omega}\eta. \tag{90}$$

The expression for the potential vorticity then reduces to

$$\zeta\equiv Hq-2\Omega\simeq\omega+\frac{\psi}{L_R^2}, \tag{91}$$

using the Rossby radius of deformation (51). The term $(1/L_R^2)\psi$ in Eq. (91) creates a shielding of the interaction between vortices (similar to the Debye shielding in plasma physics) on a length scale $\sim L_R$. In the limit $1/L_R\to0$, we recover the 2D incompressible equations. For finite L_R we can extend the statistical theory of the 2D Euler equations to the QG case by simply replacing the vorticity $\overline{\omega}$ with the potential vorticity $\overline{\zeta}$ [25–28].

VI. THE CASE OF A CIRCULAR DOMAIN OR A CHANNEL

A. Statistical equilibrium

In a disk, the angular momentum

$$L=\int h(\mathbf{r}\times\mathbf{u})_z d^2\mathbf{r} \tag{92}$$

is conserved. This additional constraint can be accounted for by adding a term $\beta\lambda\,\delta L$ in the first order variation (23). We can write $\delta L=\int\delta h(\mathbf{r}\times\mathbf{u})_z d^2\mathbf{r}+\int h(\mathbf{e}_z\times\mathbf{r})\cdot\delta\mathbf{u}d^2\mathbf{r}$, and the second term can be expressed in terms of $\delta\overline{\omega}$ and $\delta(\nabla\cdot\mathbf{u})$ by a Helmholtz decomposition of $h(\mathbf{e}_z\times\mathbf{r})$ analogous to Eq. (9), followed by an integration by part. This is analogous to the fomulas (28) and (29) used for expressing the energy variation. We can combine the energy and momentum variations by defining

$$h[\mathbf{u}-\lambda(\mathbf{e}_z\times\mathbf{r})]=-\mathbf{e}_z\times\nabla\psi'+\nabla\varphi', \tag{93}$$

instead of Eq. (9). Adding the new terms in Eqs. (31) and (33) yields the Gibbs states (35) and (41) for the velocity

$\mathbf{u}'=\mathbf{u}-\lambda(\mathbf{e}_z\times\mathbf{r})$ seen in the reference frame rotating at angular velocity λ. Accordingly, the expression of the Bernouilli function must be modified by a term of centrifugal force: we must use $B'(\psi')=gh+(\mathbf{u}'^2/2)-\lambda^2r^2$ instead of $B(\psi)$. We find therefore, that the Gibbs state (its locally averaged velocity field) is a solution of the shallow water equation, which is steady in a reference frame rotating at a modified angular velocity $\Omega+\lambda$. This velocity is indirectly determined by the constraint on angular momentum. Note that the result can be readily extended to the shallow water system on the sphere.

In the case of an annulus, the circulation $C_-=-\int u_\theta dl$ around the inner wall is an additional conserved quantity (the circulation C_+ around the outer wall is also conserved, but it is related to other conserved quantities, as $C_+=\Gamma-C_-$, and the conservation of Γ is already included in the PV conservation). Furthermore, we need in general to set $\psi=\psi_-\neq 0$ at the inner wall (while we can still set $\psi=0$ at the outer wall). As a consequence, a boundary term $-\psi_-\delta C_-$ now appears in the expression (30) for the energy variation. However, we can directly set $\delta C_-=0$, canceling this boundary term, without influence on the independent variables $h\rho$ (determining the locally averaged vorticity $\overline{\omega}=\int\sigma h\rho d\sigma$), $\boldsymbol{\nabla}\cdot\mathbf{u}$ and h. Therefore, the only modification with respect to the disk is an additional unknown ψ_- in the definition (10) of ψ, determined by the corresponding additional constraint C_-.

The case of a straight channel can be viewed as the limit of an annulus with a small width, but it can be convenient to treat it in itself. Let us consider a straight channel between two walls at coordinates $y=\pm L_y/2$ with periodic boundary conditions along the x direction (with domain length L_x). In the absence of Coriolis force ($\Omega=0$), the x-wise momentum

$$P=\int hu_x d^2\mathbf{r} \tag{94}$$

is conserved (instead of the angular momentum), as well as the circulation $C_+=-\int u_x dx$ along the upper wall ($y=L_y/2$). The boundary condition (10) defining ψ is replaced by $\psi=P/L_x$ at the upper wall $y=L_y/2$ and $\psi=0$ (for instance) at the lower wall $y=-L_y/2$. Unlike with angular momentum, we cannot express the variation δP in terms of the variations in the independent variables $\overline{\omega},\boldsymbol{\nabla}\cdot\mathbf{u},h$. However, we have now an additional freedom in the variational problem, as we can add a uniform x-wise velocity $-U\mathbf{e}_x$ (use a reference frame with velocity $U\mathbf{e}_x$) without influence on the independent variables $\overline{\omega},\boldsymbol{\nabla}\cdot\mathbf{u}$, h. For any choice of U, we can solve the variational problem with the velocity $\mathbf{u}'=\mathbf{u}-U\mathbf{e}_x$, corresponding energy $E'=E+MU^2/2-PU$, and upper wall circulation $C'_+=C_+-UL_x$. This yields a Gibbs state (35),(38),(41) representing a steady solution of the shallow water equation in a reference frame moving with velocity $U\mathbf{e}_x$. Among these states, the ones with the right value of the momentum $P=\int hu_x d^2\mathbf{r}$ will be the actual solutions. Families of Gibbs states with the same structure translated in the x direction are obtained, as discussed by Sommeria *et al.* [18] in the incompressible case.

Finally, in the case of an infinite domain, the two components of momentum, as well as the angular momentum are conserved. This yields to purely translating or purely rotating Gibbs states, as discussed by Chavanis and Sommeria [44] in the incompressible case.

B. Relaxation equations

Taking the derivative of Eqs. (94) and (92) with respect to time and using Eqs. (58) and (59), we obtain the constraints

$$\dot{P}=\int hJ_{\omega y}d^2\mathbf{r}=0, \tag{95}$$

$$\dot{L}=-\int h\mathbf{J}_\omega\cdot\mathbf{r}d^2r=0. \tag{96}$$

These constraints can be included in the variational principle (67) by introducing appropriate Lagrange multipliers denoted as $\beta(t)U(t)$ and $\beta(t)\lambda(t)$. Then, the results of Sec. V are generalized simply by replacing the velocity $\overline{\mathbf{u}}$ by the relative velocity $\overline{\mathbf{u}'}=\overline{\mathbf{u}}-U(t)\mathbf{e}_x-\lambda(t)\mathbf{e}_z\times\mathbf{r}$ where the time evolution of $U(t)$ and $\lambda(t)$ is obtained by substituting the optimal current (73), with the above transformation, in constraints (95) and (96). In the case of a channel, we have the additional conserved quantity $C_+=-\int u_x dx$ along the upper wall. Using Eq. (59), we readily find that $\dot{C}_+=\int J_{\omega y}dx=0$ as the current $\mathbf{J}_\omega$ is parallel to the wall, so the circulation along each wall is indeed conserved by the relaxation equations.

VII. CONCLUSION

We have applied equilibrium statistical mechanics to predict the self-organization of the shallow water system, assuming that the velocity divergence as well as the height field h remain smooth, while vorticity undergoes filamentation into fine-scale structures. This regime is expected in the absence of shocks, typically for flows submitted to a strong Coriolis effect, for which our present approach generalizes the earlier QG (incompressible) results. We predict an organization into a steady flow, after smoothing of the fine-scale vorticity fluctuations. It is characterized by a particular monotonic relationship between PV and the stream function ψ, defined by Eq. (12). It is a balanced motion, reducing to the geostrophic balance (49) for unidirectional solutions and to the cyclostrophic balance (57) for axisymmetric ones. In other words, the wave mode φ vanishes. In domains with symmetries by translation or rotation, steadily translating or rotating solutions are obtained (see Sec. VI). It is remarkable that these dynamical properties emerge from entropy maximization. Moreover, the Boltzmann entropy (21) is the only expression [of the general form (22)] that yields a solution to the maximization problem, unlike in the Euler case.

The predicted state appears as the most likely result of random PV reorganization (stirring) taking into account energy conservation. This provides a general explanation for the emergence of such steady flow structures—jets or isolated vortices—a phenomenon widely observed in natural systems and numerical simulations. Practical predictions can be obtained by solving the system (42) and (43), reducing to Eq. (52) in the unidirectional case. It is, however, more convenient in general to solve relaxation equations as developed in Sec. V. It involves modified shallow water equations (58), (60), and (74) and a relaxation equation (62) with a current (69) for each PV level. The case of two PV levels is explicited with more details in Sec. V C. For the purpose of calculating the equilibrium states, the advective terms of these equations (on the left-hand side) can be dropped out, and the diffusion coefficient D is arbitrary (but positive). The time evolution relaxes towards a Gibbs state, and it selects a true entropy maximum among the different solutions of Eqs. (42) and (43), which are just critical points of entropy.

These predictions correspond to an ideal property of PV mixing with the only constraint of the conservation laws. The system may not actually reach such equilibrium by free evolution for various reasons, as discussed in the Introduction. In particular, systems of geophysical interest are permanently forced instead of freely decaying. However, we can still use the relaxation equations as subgrid-scale models for a time evolution simulation of the explicitly resolved scales. The idea is that eddy fluxes will tend to drive the system toward the statistical equilibrium. Then the diffusion coefficient has to be adjusted to ''realistic'' values [27].

This subgrid-scale modeling involves a term of PV eddy diffusion and a *forcing* or drift term. The latter is reminiscent of the ''neptune'' effect—an intriguing forcing effect proposed by Holloway [32] as a driving of a mean flow by random geostrophic turbulence in the presence of bottom topography. This was obtained by another statistical mechanics approach for truncated spectral models of the QG system, but it can be viewed as a particular limit of the present approach [27]. This forcing here naturally appears in association with the eddy diffusion of PV, as the result of the energy conservation constraint in the maximum entropy production procedure. Note that more direct justification of this term has been also obtained (in the incompressible Euler case) in terms of kinetic models by Chavanis [48–50]. In this point of view, the drift term is a result of a polarization process. These kinetic models yield more complex expressions of the eddy fluxes, involving integral over previous times, which reduce to our MEP expression only close to an equilibrium state. Moreover, as discussed in Refs. [53] and [54], the assumption of a global energy constraint (65) should be replaced with a more local condition in large systems, and neglecting the energy of subgrid-scale fluctuations is appropriate only when the cutoff is much smaller than the Rossby radius of deformation. However, the fundamental existence of a drift term, a ''generalized neptune effect'' in addition to PV eddy diffusion, is confirmed by all these approaches. This allows one to incorporate the fruitfull idea of PV uniformization in operational modeling. The principle of extension to multilayer shallow water systems, such as the model MICOM [34] used in oceanography is straightforward.

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